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Method of Action–Angle Variables and the Classical Dynamics of a Nonlinear Lagrangian

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Abstract

The method of action–angle variables is used to obtain the complete periodic solutions of a nonlinear chiral Lagrangian system with the Lagrangian of the form $\mathscr{L} = \frac{1}{2}\{\dot{\mathbf{q}}^2 + [\lambda(\mathbf{q}\cdot\dot{\mathbf{q}})^2/(1-\lambda\mathbf{q}^2)] - [k_0\mathbf{q}^2/(1-\lambda\mathbf{q}^2)]\}\ \mathbf{q} = (q_1, q_2, q_3)$ by making suitable canonical transformations. Usual semiclassical quantization procedure may then be applied to obtain the energy levels, which is shown to be in good agreement with exact results.

1. *Introduction*

The method of action–angle variables is not only an esoteric technique of classical mechanics but a powerful tool in the understanding of periodic solutions of classical dynamical systems, and further it is a "royal road to quantization" as Sommerfeld puts it. Increased interest has been evinced recently in this method, as one observes from its usage in the studies of the motion of the Morse oscillator in chemical physics (Porter et al., 1975) and anharmonic oscillator (Mathews and Eswaran, 1972) in quantum mechanics, coupled oscillator systems perturbed by nonlinear nearest-neighbor interactions (Ford, 1974) in statistical mechanics, solution of nonlinear evolution equations by inverse method (wherein the method is interpreted as a canonical transformation to action–angle variables, see for example McLaughlin, 1975) and quantization of the Sine–Gordon equation by the action–angle variable method (Faddeev, 1975). It is the aim of the present paper to show that the classical dynamics and in particular the bounded motions of a nonlinear chiral Lagrangian of the form

$$\mathscr{L} = \frac{1}{2}\left[\dot{\mathbf{q}}^2 + \frac{\lambda(\mathbf{q}\cdot\dot{\mathbf{q}})^2}{(1-\lambda\mathbf{q}^2)} - \frac{k_0\mathbf{q}^2}{(1-\lambda\mathbf{q}^2)}\right] \qquad (\mathbf{q} = q_1, q_2, q_3) \tag{1.1}$$

may be analyzed explicitly by the method of action–angle variables. The above system in the $k_0 = 0$ limit is a particular parametrization of the $SU(2) \times SU(2)$ Lagrangian [and is equivalent to the zero-space dimensional version of the Lagrangian of equation (3.39), Gasiorowicz and Geffen, 1969]. The exact quantization of this system was discussed recently by us (Lakshmanan and Eswaran, 1975). Here our main purpose is to show the complete analyzability of the classical bounded motions by the method of action–angle variables.

The plan of the paper is as follows. In Section 2 we give a brief discussion of the system Hamiltonian and in Section 3 we perform the canonical transformation to the action–angle variables. Section 4 contains the complete periodic solutions of the system, while in Section 5 we give a brief discussion of the semiclassical quantization of the system.

2. *The System Hamiltonian*

The canonically conjugate momentum corresponding to the Lagrangian is

$$\mathbf{p} = \frac{\partial\mathscr{L}}{\partial\dot{\mathbf{q}}} = \frac{\dot{\mathbf{q}}}{(1-\lambda\mathbf{q}^2)} \tag{2.1}$$

whence the Hamiltonian becomes

$$H = E = \frac{1}{2}\left[\dot{\mathbf{q}}^2 + \frac{\lambda(\mathbf{q}\cdot\dot{\mathbf{q}})^2}{(1-\lambda\dot{\mathbf{q}}^2)} + \frac{k_0\mathbf{q}^2}{(1-\lambda\mathbf{q}^2)}\right] \tag{2.2}$$

On expressing (2.2) in terms of spherical polar coordinates (q, θ, ϕ) the Hamiltonian becomes

$$H = E = \frac{1}{2}\left[\frac{\dot{\mathbf{q}}^2 + k_0 q^2}{1-\lambda q^2} + \frac{L^2}{q^2}\right] \tag{2.3}$$

where

$$L^2 = q^4\,[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2] = \text{const} \tag{2.4}$$

Solving for q at a fixed energy E gives

$$\dot{q} = \pm\frac{1}{q}\left[\frac{(b-a)}{2}\,q^4 + \left(\frac{b+a}{2\lambda}\right)q^2 + c\right] \tag{2.5}$$

where

$$a = 4E\lambda + k_0 + \lambda^2 L^2$$
$$b = -k_0 + \lambda^2 L^2 \tag{2.6}$$

and

$$c = -L^2$$

3. *Transformation to Action-Angle Variables*

From the general theory of canonical transformation in classical mechanics it is known (Goldstein, 1950) that when the Hamiltonian is not explicitly time-dependent, and the system is periodic, the suitable method to analyze the dynamical problem is to transform to a system of conjugate variables (Q_i, P_i) in which the new momenta P_i are action variables and Q_i are cyclic coordinates. The generator that induces such a transformation is Hamilton's characteristic function (Goldstein, 1950),

$$W(q, \theta, \phi; P_1, P_2, P_3) = W_q + W_\theta + W_\phi \tag{3.1}$$

where

$$W_q = \int p_q\, dq = \int \frac{\dot{q}}{(1 - \lambda q^2)}\, dq \tag{3.2a}$$

$$W_\theta = \int p_\theta\, d\theta = \int \left(L^2 - \frac{M^2}{\sin^2\theta}\right)^{1/2} d\theta \tag{3.2b}$$

and

$$W_\phi = \int p_\phi d\phi = M\phi \tag{3.2c}$$

Then the equations of motion in terms of the new variables become

$$\dot{Q}_i = \frac{\partial H}{\partial P_i} = \nu_i \qquad \text{(const)} \tag{3.3a}$$

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} = 0 \tag{3.3b}$$

the new and old Hamiltonians being identical. We note that

$$Q_i = \nu_i t + \beta_i \tag{3.4a}$$

and

$$P_i = J_i \qquad \text{(constants)} \tag{3.4b}$$

The idea is then to choose the P_i's such that they are constants of motion and calculate Q_i from the relation

$$Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial J_i} \tag{3.5}$$

Comparison of (3.5) with (3.4a) gives the necessary solution. We then proceed as shown below.

The first of the three conserved momenta that are proportional to the action variables in our case (2.2) is

$$P_1 \equiv N = \frac{1}{2\pi}\oint p_q\, dq$$

$$= \frac{1}{4\pi}\int_{\xi_<}^{\xi_>}\left[c + \left(\frac{b+a}{2}\right)\xi + \left(\frac{b-a}{2}\right)\xi^2\right]^{1/2}\frac{d\xi}{(1-\lambda\xi)\xi} \qquad (3.6)$$

when we have made the substitution $q^2 = \xi$ in the integral (3.2a). Since

$$\xi_> = \frac{-(b+a) - [(b+a)^2 - 8(b-a)c\lambda^2]}{2\lambda(b-a)} \qquad (3.7a)$$

and

$$\xi_< = \frac{-(b+a) + [(b+a)^2 - 8(b-a)c\lambda^2]}{2\lambda(b-a)} \qquad (3.7b)$$

we have

$$P_1 \equiv N = \frac{-(-c)^{1/2}}{2} + \frac{(a-b)^{1/2}}{2(2)^{1/2}\lambda} - \frac{[-(c\lambda^2+b)]^{1/2}}{2\lambda} \qquad (3.8)$$

The other two are

$$P_2 \equiv L = \left(p_\theta{}^2 + \frac{p_\phi{}^2}{\sin^2\theta}\right)^{1/2} \qquad (3.9)$$

and

$$P_3 \equiv M = p_\phi \qquad (3.10)$$

Then from (3.5) and using the quantities given in the Appendix we evaluate the new coordinates as follows:

$$Q_N = \left(\frac{\partial W}{\partial N}\right)_{L,M} = \left(\frac{\partial W_q}{\partial a}\right)_{b,c}\Bigg/\left(\frac{\partial N}{\partial a}\right)_{b,c}$$

$$= \sin^{-1}\left[\frac{a+b+2\lambda(b-a)\xi}{R^{1/2}}\right] \qquad (3.11)$$

$$Q_L = \left(\frac{\partial W}{\partial L}\right)_{N,M} = -\left(\frac{\partial W}{\partial a}\right)_{b,c}\left[\left(\frac{\partial N}{\partial b}\right)_{a,c}\frac{db}{dL} + \left(\frac{\partial N}{\partial c}\right)_{a,b}\frac{dc}{dL}\right]\Bigg/\left(\frac{\partial N}{\partial a}\right)_{b,c}$$

$$+\left(\frac{\partial W_q}{\partial b}\right)_{a,c}\left(\frac{db}{dL}\right) + \left(\frac{\partial W_q}{\partial c}\right)\left(\frac{dc}{dL}\right) + \left(\frac{\partial W_\theta}{\partial L}\right)_M$$

$$= q_L - \frac{\lambda L}{2}\left(\frac{2}{a-b}\right)^{1/2}\sin^{-1}\left[\frac{(a+b)+2\lambda(b-a)\xi}{R^{1/2}}\right]$$

$$+\frac{L}{2(-c)^{1/2}}\left\{\sin^{-1}\left[\frac{(a+b)+2\lambda(b-a)\xi}{R^{1/2}}\right] + \sin^{-1}\left[\frac{(a+b)\xi+4c\lambda}{\xi R^{1/2}}\right]\right\}$$

$$+\frac{\lambda^2 L}{2[-(b+c\lambda^2)]^{1/2}}\sin^{-1}\left[\frac{(1-\lambda\xi)(a-3b)+4(b+c\lambda^2)}{(1-\lambda\xi)R^{1/2}}\right]$$

$$+\frac{\lambda L}{2[-(b+c\lambda^2)]^{1/2}}\sin^{-1}\left[\frac{(1-\lambda\xi)(a+b+4c\lambda^2)+4\lambda\xi(b+c\lambda^2)}{(1-\lambda\xi)R^{1/2}}\right]$$

where

$$R = (a+b)^2 + 8\lambda^2(\dot{a}-b)c \tag{3.12}$$

$$Q_M = \left(\frac{\partial W}{\partial M}\right)_{N,L} = \left(\frac{\partial W_\theta}{\partial M}\right)_L + \left(\frac{\partial W_\phi}{\partial M}\right)$$

$$= \phi - \cos^{-1}\left[\frac{\eta\cot\theta}{(1-\eta^2)^{1/2}}\right], \qquad \eta = \frac{M}{L} \tag{3.13}$$

In equations (3.11)-(3.13) the old coordinates (ξ, θ, ϕ) are held constant in all differentiations. Now using the equations of motion, we find from (2.6) and (3.8)-(3.10) that

$$\dot{Q}_N = \left(\frac{\partial E}{\partial N}\right)_{L,M} = 2\left[\frac{(a-b)}{2}\right]^{1/2} = 2(k_0 + 2E\lambda)^{1/2} \equiv \omega_N \tag{3.14a}$$

$$\dot{Q}_L = \left(\frac{\partial E}{\partial L}\right)_{N,M} = \frac{L}{(-c)^{1/2}}\left(\frac{a-b}{2}\right)^{1/2} = \frac{\omega_N}{2} \equiv \omega_L \tag{3.14b}$$

and

$$\dot{Q}_M = \left(\frac{\partial E}{\partial M}\right)_{N,L} = 0 \equiv \omega_M \tag{3.14c}$$

Thus the solution of the equation of motion takes the rather simple form

$$Q_N(t) = \delta_N + \omega_N t \tag{3.15a}$$

$$Q_L(t) = \delta_L + \omega_L t \tag{3.15b}$$

$$Q_M(t) = \delta_M \tag{3.15c}$$

4. *The Complete Periodic Solution*

The radial orbit is easily obtained from (3.11) and (3.15a):

$$q(t) = A\left[1 - \beta \sin^2\left(\frac{\omega_N t}{2} + \zeta\right)\right]^{1/2} \tag{4.1}$$

where

$$A = \left[\frac{R^{1/2} - (a+b)}{2\lambda(b-a)}\right]^{1/2} \tag{4.2}$$

$$\beta = \frac{2R^{1/2}}{R^{1/2} - (a+b)} \tag{4.3}$$

$$(|A| \leqslant \lambda^{-1/2} \qquad \text{when} \qquad \lambda > 0)$$

and $\zeta = \delta_N/2 - \pi/4$. One may substitute (4.1)-(4.3) in (2.3) and verify that $H = E$. Similarly the orbit for θ is obtained from equations (3.12) and (3.15b) to be

$$\cos\theta(t) = (1 - \eta^2)^{1/2} \cos\left[\omega_L t + \delta_L + L\Delta_L\right] \tag{4.4}$$

where

$$\begin{aligned}\Delta_L = {} & \frac{\lambda}{[2(a-b)]^{1/2}} \sin^{-1}\left[\frac{a+b+2\lambda(b-a)\chi}{R^{1/2}}\right] \\ & - \frac{1}{2(-c)^{1/2}}\left\{\sin^{-1}\left[\frac{a+b+2\lambda(b-a)}{R^{1/2}}\right] + \sin^{-1}\left[\frac{(a+b)\chi + 4c\lambda}{\chi R^{1/2}}\right]\right\} \\ & - \frac{\lambda^2}{2[-(b+c\lambda^2)]^{1/2}} \sin^{-1}\left[\frac{(1-\lambda)(a-3b) + 4(b+c\lambda^2)}{1-R^{1/2}}\right] \\ & - \frac{\lambda}{2[-(b+c\lambda^2)^{1/2}} \sin^{-1}\left[\frac{(1-\lambda)(a+b+4c\lambda^2) + 4(b+c\lambda^2)}{1-R^{1/2}}\right]\end{aligned} \tag{4.5}$$

and

$$\chi = [2\lambda(b-a)]^{-1}\{-(a+b) + R^{1/2}\sin(\omega_N t + \delta_N)\} \tag{4.6}$$

Finally from equations (3.13) and (3.15c) we have

$$\phi(t) = \delta_M + \cos^{-1}\left[\frac{\eta \cot\theta(t)}{(1-\eta^2)^{1/2}}\right] \tag{4.7}$$

With the aid of equations (4.1)–(4.7) we may express the Cartesian coordinates and momenta in terms of the variables Q_N, Q_L, and Q_M and thus through equations (3.14) as functions of t and of the constant angular momenta. Transformation to the polar coordinates

$$\begin{aligned} x &= q_1 = q \sin\theta \cos\phi \\ y &= q_2 = q \sin\theta \sin\phi \\ z &= q_3 = q \cos\theta \end{aligned} \tag{4.8}$$

gives the conjugate momenta

$$\begin{aligned} p_q &= p_x \sin\theta \cos\phi + p_y \sin\theta \sin\phi + p_z \cos\theta \\ p_\theta &= p_x q \cos\theta \cos\phi + p_y q \cos\theta \sin\phi + p_z q \cos\theta \\ p_\phi &= -p_x q \sin\theta \sin\phi + p_y q \sin\theta \cos\phi \end{aligned} \tag{4.9}$$

Making the inverse transformation we have in matrix notation

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} p_q \\ p_\theta/q \\ p_\phi/q\sin\theta \end{bmatrix} \tag{4.10}$$

Thus we have

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = q(t) \begin{bmatrix} -\sin q_L \sin Q_M + \eta \cos q_L \cos Q_M \\ \sin q_L \cos Q_M + \eta \cos q_L \cos Q_M \\ (1-\eta^2)^{1/2} \cos q_L \end{bmatrix} \tag{4.11}$$

and

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = p_q \begin{bmatrix} -\sin q_L \sin Q_M + \eta \cos q_L \cos Q_M \\ \sin q_L \cos Q_M + \eta \cos q_L \sin Q_M \\ +(1-\eta^2)^{1/2} \cos q_L \end{bmatrix}$$

$$+ \frac{L}{q} \begin{bmatrix} -\cos q_L \sin Q_M - \eta \sin q_L \cos Q_M \\ \cos q_L \cos Q_M - \eta \sin q_L \sin Q_M \\ -(1-\eta^2)^{1/2} \sin q_L \end{bmatrix} \tag{4.12}$$

where

$$q_L = Q_L + L\Delta_L \tag{4.13}$$

5. *Discussion*

We have shown that the bounded periodic solutions of a nonlinear chiral Lagrangian may be obtained by the method of action-angle variables. By the usual procedure of Bohr-Sommerfeld the canonical action variables may be quantized (with slight modification):

$$N = (n_r + \tfrac{1}{2})\hbar \tag{5.1a}$$

$$L = (l + \tfrac{1}{2})\hbar \tag{5.1b}$$

and

$$M = m\hbar \tag{5.1c}$$

to obtain the energy level expression as

$$E_{n_r, l} = (2n_r + l + \tfrac{3}{2})k_0^{1/2}\hbar + \tfrac{1}{2}\lambda(2n_r + l + \tfrac{3}{2})^2\hbar^2 \tag{5.2}$$

which is in agreement with the exact quantized energy level expressions (Lakshmanan and Eswaran, 1975) apart from the constant factor. It might now be interesting to see whether the actual (3 + 1)-dimensional field Hamiltonian itself may be expressed—in terms of action-angle variables—by making suitable canonical transformation as is the case of the Sine-Gordon field case.

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Appendix

In this Appendix we give the necessary quantities to evaluate the Q_i's:

$$\frac{\partial W_q}{\partial a} = \frac{1}{8\lambda}\left(\frac{2}{a-b}\right)^{1/2} \sin^{-1}\left[(a+b) + 2\lambda(b-a)\xi\right]/R^{1/2} \tag{A.1}$$

$$\frac{\partial W_q}{\partial b} = \frac{1}{4[-(b+c\lambda^2)]^{1/2}} \sin^{-1}\left[\frac{(1-\lambda\xi)(a-3b) + 4(b+c\lambda^2)}{(1-\lambda\xi)R^{1/2}}\right]$$

$$- \frac{1}{4\lambda}\left(\frac{2}{a-b}\right)^{1/2} \sin^{-1}\left[\frac{(a+b) + 2\lambda(b-a)\xi}{R^{1/2}}\right] \tag{A.2}$$

$$\frac{\partial W_q}{\partial c} = -\frac{1}{4(-c)^{1/2}} \sin^{-1}\left[\frac{(a+b)\xi + 4c\lambda}{R}\right] + \frac{\lambda}{4[-(b+c\lambda^2)]^{1/2}} \sin^{-1}\left[\frac{(1-\lambda\xi)(4c\lambda^2 + a + b) + 4\lambda\xi(b + c\lambda^2)}{(1-\lambda\xi)R^{1/2}}\right] \tag{A.3}$$

where

$$R = (a+b)^2 + 8(a-b)c\lambda^2$$

$$\frac{\partial W_\theta}{\partial L} = q_L = \cos^{-1}\left(\frac{\cos\theta}{1-\eta^2}\right) \tag{A.4}$$

and

$$\frac{\partial W_\theta}{\partial M} = -\cos^{-1}\left[\frac{\eta}{(1-\eta^2)^{1/2}}\cos\theta\right] \tag{A.5}$$

where

$$\eta = M/L \tag{A.6}$$

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